

Quasi-actions on trees

Research announcement

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We present a battery of techniques for investigating quasi-isometric rigidity of graphs of groups [MSW]. The techniques work best when all edge and vertex groups are “coarse Poincaré duality” groups in the sense of [KK99], for example, fundamental groups of closed, aspherical manifolds—such groups respond well to analysis using methods of coarse algebraic topology introduced in [FS96] and further developed in [KK99]. Our techniques also require the Bass-Serre tree to be bushy, meaning that it has infinitely many ends. For example, when the dimensions of the vertex and edge groups are homogeneous we obtain:

Theorem 1 (Homogeneous Theorem). *Fix an integer $n \geq 0$, and let Γ be a finite graph of coarse PD(n) groups with bushy Bass-Serre tree. Let H be a finitely generated group quasi-isometric to $\pi_1\Gamma$. Then H is the fundamental group of a graph of groups with bushy Bass-Serre tree, and with vertex and edge groups quasi-isometric to those of Γ .*

Specializing to $n = 0$ gives a theorem of Stallings–Dunwoody [Sta68], [Dun85] which says, in modern language, that if a finitely generated group H is quasi-isometric to a free group then H is the fundamental group of a finite graph of finite groups, implying furthermore that H is virtually free.

Theorem 1 suggests the following problem. Given Γ as above, all edge-to-vertex group injections must have finite index image, and so all vertex and edge groups lie in a single quasi-isometry class. Given \mathcal{C} , a quasi-isometry class of coarse PD(n) groups, let $\Gamma(\mathcal{C})$ be the class of fundamental groups of finite graphs of groups with vertex and edge groups in \mathcal{C} and with bushy Bass-Serre tree. Theorem 1 says that $\Gamma(\mathcal{C})$ is closed up to quasi-isometry.

Problem 2. *Given \mathcal{C} , describe the quasi-isometry classes within $\Gamma(\mathcal{C})$.*

Here’s a rundown of the cases for which we know the solution to this problem. In the case $n = 1$, where $\mathcal{C} = \{\text{virtually } \mathbf{Z}\}$: the amenable groups in $\Gamma(\mathcal{C})$ are classified in [FM98], and the nonamenable ones in [Why99]. For $\mathcal{C} = \{\text{virtually } \mathbf{Z}^n\}$,

the amenable groups in $\Gamma(\mathcal{C})$ are classified in [FM00b]; the nonamenable case remains open. For $\mathcal{C} = \{\text{quasi-isometric to } \mathbf{H}^2\} = \{\text{finite-by-(cocompact fuchsian)}\}$, the subclass of $\Gamma(\mathcal{C})$ consisting of word hyperbolic surface-by-free groups is quasi-isometrically rigid and is classified in [FM00a], but the broader classification in $\Gamma(\mathcal{C})$ is open.

If \mathcal{C} is the quasi-isometry class of cocompact lattices in an irreducible, semisimple Lie group L with finite center, then combining Mostow Rigidity for L with quasi-isometric rigidity (see [Far97] for a survey) it follows that for each $G \in \mathcal{C}$ there exists a homomorphism $G \rightarrow L$ with finite kernel and discrete, cocompact image, and this homomorphism is unique up to post-composition with an inner automorphism of L . Combining this with Theorem 1 it follows that any group in $\Gamma(\mathcal{C})$ is quasi-isometric to the cartesian product of any group in \mathcal{C} with any free group of rank ≥ 2 .

Our techniques also apply to graphs of coarse PD groups without constant dimension, under various assumptions on how edge spaces attach to the vertex spaces. Here is a sample application:

Theorem 3. *Let Γ be a graph of finitely generated abelian groups. Suppose that the following condition holds:*

- (*) *for each vertex group Γ_v of dimension n , all edge groups incident to Γ_v all have dimension $< n$, and if there exist any incident edge groups of dimension $n - 1$ then the incident edge groups span Γ_v .*

Then any finitely generated group H quasi-isometric to Γ is the fundamental group of a graph of virtually abelian groups.

Sometimes we can combine our techniques with other results to get even stronger quasi-isometric rigidity theorems. For example, by using the Geodesic Pattern Rigidity Theorem of R. Schwartz ([Sch97], see Theorem 13 below) we obtain:

Theorem 4. *Let Γ be a finite graph of groups whose vertex groups are fundamental groups of closed hyperbolic manifolds of dimension ≥ 3 , and whose edge groups are infinite cyclic. If H is a finitely generated group quasi-isometric to $\pi_1\Gamma$, then H is the fundamental group of a graph of groups Γ' , such that each vertex group of Γ' is weakly commensurable¹ either to a vertex group of Γ or to \mathbf{Z} , and such that each edge group of Γ' is commensurable to \mathbf{Z} .*

We will also find a way to strengthen Theorem 3 by developing Abelian Pattern Rigidity (Theorem 14).

¹ *Weak commensurability* of groups G_1, G_2 means that there exists a group Q and homomorphisms $Q \rightarrow G_1, Q \rightarrow G_2$ each with finite kernel and finite index image. This is the smallest equivalence relation incorporating passage both to a finite index subgroup and to a finite kernel quotient.

And there is more to the conclusions of Theorems 3 and 4 than is stated: in each case the new graph of groups not only has edge and vertex groups related to those of the old graph of groups, but the edge-to-vertex injections are also related. This will be made explicit in Theorem 11 and in the applications of Theorem 11 to the proofs of Theorems 3 and 4.

Remark Theorems 3 and 4 are just samples. The main result on Inhomogeneous Rigidity, Theorem 11, has a multitude of applications. In this research announcement we give only sketches of proofs, and we focus particularly on coarse PD vertex and edge groups, ignoring wider contexts for our results. Full statements in wider contexts, and full proofs, will be found in [MSW].

Remark As in the homogeneous case, the techniques for the inhomogeneous case can be applied to quasi-isometric classification as well as to quasi-isometric rigidity; although we have not mentioned here any of these classification theorems, the preprint [MSW] will include some results. We should also mention [PW00], where techniques similar to those of Theorem 11 are used in proving that the quasi-isometric classification of accessible groups completely reduces to the classification of one-ended groups.

Graphs of groups and Bass-Serre trees: a review

A graph of groups Γ of finite type consists of the following data. Start with a finite graph Γ with vertex set $\mathcal{V}(\Gamma)$ and edge set $\mathcal{E}(\Gamma)$. For each edge $e \in \mathcal{E}(\Gamma)$, the two ends of e form a set $\text{Ends}(e)$, and each end $\eta \in \text{Ends}(e)$ is attached to some vertex $v(\eta) \in \mathcal{V}(\Gamma)$. Associated to each vertex $v \in \mathcal{V}(\Gamma)$ there is a finitely generated vertex group Γ_v , associated to each edge $e \in \mathcal{E}(\Gamma)$ there is an edge group Γ_e , and associated to each end $\eta \in \text{Ends}(e)$ there is an edge-to-vertex injection $\gamma_\eta: \Gamma_e \rightarrow \Gamma_{v(\eta)}$.

Associated to a graph of groups Γ is a *graph of spaces* B_Γ , as follows. Choose path connected, pointed spaces $B(v)$, $v \in \mathcal{V}(\Gamma)$, and $B(e)$, $e \in \mathcal{E}(\Gamma)$, whose fundamental groups are identified with the associated vertex or edge groups of Γ . Choose pointed *attaching maps* $\xi_\eta: B(e) \rightarrow B(v)$ inducing the injections γ_η . For each edge e let $\hat{e} = \text{int}(e) \cup \text{Ends}(e)$ denote the end compactification of e , (\hat{e} is a compact arc, regardless of whether e is a compact arc or a loop). Construct a quotient space B_Γ from the disjoint union of the set

$$\{B(v), B(e) \times \hat{e} \mid v \in \mathcal{V}(\Gamma), e \in \mathcal{E}(\Gamma)\}$$

by gluing $B(e) \times \eta$ to $B(v(\eta))$ via the map $(x, \eta) \rightarrow \xi_\eta(x)$, for each $e \in \mathcal{E}(\Gamma)$, $\eta \in \text{Ends}(e)$. The fundamental group $\pi_1(B_\Gamma)$ is well-defined up to isomorphism, and it is

called the fundamental group of Γ , denoted $\pi_1(\Gamma)$.

The map $B \rightarrow \Gamma$, taking $B(v)$ to v and projecting $B(e) \times \hat{e}$ to e , induces a decomposition of B into point preimages. In the universal cover $X = \tilde{B}$, taking connected lifts via the universal covering map $X \rightarrow B$ of point preimages of $B \rightarrow \Gamma$ gives a decomposition of X into path connected sets. This decomposition of X is $\pi_1(\Gamma)$ -equivariant. The quotient space of this decomposition of X is a tree T on which $\pi_1(\Gamma)$ acts, the *Bass-Serre tree* of Γ . This action is well-defined up to equivariant tree isomorphisms, independent of the choices, and the quotient graph of T is canonically identified with Γ :

$$\begin{array}{ccc} X & \longrightarrow & T \\ \downarrow & & \downarrow \\ B & \longrightarrow & \Gamma \end{array}$$

The map $X \rightarrow T$ is called the *Bass-Serre tree of spaces* associated to the graph of spaces $B \rightarrow \Gamma$. The inverse image of a vertex $v \in T$ is a *vertex space* $X(v)$ of X . The inverse image of the midpoint of an edge e of T is called an *edge space* $X(e)$ of X . The topological space X is constructed from the disjoint union of the set $\{X(v), X(e) \times e \mid v \in \mathcal{V}(T), e \in \mathcal{E}(T)\}$, where for each vertex v and edge e incident to v we glue $X(e) \times v$ to a subset of $X(v)$ via an attaching map $X(e) \rightarrow X(v)$ which is a lift of an attaching map for the graph of spaces B . The image of the attaching map $X(e) \rightarrow X(v)$ is called an *incident edge space inside* $X(v)$. The set of incident edge spaces inside $X(v)$ is called the *edge space pattern inside* $X(v)$.

A simple trichotomy holds for Bass-Serre trees: T is *bounded*; or T is *line-like* meaning that it contains a line as a cobounded subset; or T is *bushy* meaning that it has infinitely many ends. A graph of groups Γ (and its associated Bass-Serre tree) is said to be *reduced* if for each vertex v , the number of surjective edge-to-vertex injections $\gamma_\eta: \Gamma_e \rightarrow \Gamma_v$ is not equal to 1; it can be 0 or ≥ 2 . When T is reduced, the trichotomy simplifies as follows. First, T is bounded if and only if T and Γ are each a point. Second, T is line-like if and only if T is a line and Γ is a *mapping torus*, meaning either a circle with isomorphic edge-to-vertex inclusions all around, or an arc with isomorphic inclusions in the interior and index 2 inclusions at the endpoints. Finally, T is bushy if and only if T has at least one vertex of valence ≥ 3 ; valence of a vertex in T is easily computed in terms of the image vertex in Γ , as the sum of the indices of the edge groups inside the vertex group.

Coarse language

Let X be a metric space. Given $A \subset X$ and $R \geq 0$, denote $N_R(A) = \{x \in X \mid \exists a \in A \text{ such that } d(a, x) \leq R\}$. Given subsets $A, B \subset X$, let $A \subset_c B [R]$ denote

$A \subset N_R(B)$. Let $A \subset_c B$ denote the existence of $R \geq 0$ such that $A \subset_c B [R]$; this is called *coarse containment* of A in B . Let $A =_c B [R]$ denote $A \subset_c B [R]$ and $B \subset_c A [R]$. Let $A =_c B$ denote the existence of R such that $A =_c B [R]$; this is called *coarse equivalence* of A and B .

Given a metric space X and subsets A, B , we say that a subset C is a *coarse intersection* of A and B if for all sufficiently large R we have $N_R(A) \cap N_R(B) =_c C$. A coarse intersection of A and B may not exist, but if one does exist then it is well-defined up to coarse equivalence.

Given metric spaces X, Y , a map $f: X \rightarrow Y$ is a *uniform embedding* if there exists proper, increasing functions $g, h: [0, \infty) \rightarrow [0, \infty)$ such that

$$g(d_X(x, y)) \leq d_Y(fx, fy) \leq h(d_X(x, y))$$

If X, Y are geodesic metric spaces then the upper bound h can always be taken to be an affine function. When $h(d) = Kd + C$ and $g(d) = \frac{1}{K}d - C$ then we say that f is a K, C *quasi-isometric embedding*. If this is so then we say in addition that f is a K, C *quasi-isometry from X to Y* if $f(X) =_c Y [C]$. A C' -*coarse inverse* of f is a K, C' quasi-isometry $g: Y \rightarrow X$ such that $x =_c g(f(x)) [C']$ and $y =_c f(g(y)) [C']$, for all $x \in X, y \in Y$. A simple fact says that for all K, C there exists C' such that each K, C quasi-isometry has a C' -coarse inverse.

Let G be a group and X a metric space. A K, C *quasi-action* of G on X is a map $(g, x) \mapsto g \cdot x$ from $G \times X$ to X , such that: for each g the map $x \mapsto g \cdot x$ is a K, C quasi-isometry; and for each $x \in X, g, h \in G$ we have

$$g \cdot (h \cdot x) =_c (gh) \cdot x [C]$$

A quasi-action is *cobounded* if there exists a constant R such that for each $x \in X$ we have $G \cdot x =_c X [R]$. A quasi-action is *proper* if for each R there exists M such that for all $x, y \in X$, the cardinality of the set $\{g \in G \mid (g \cdot N(x, R)) \cap N(y, R) \neq \emptyset\}$ is at most M .

A fundamental principle of geometric group theory says that if G is a finitely generated group equipped with the word metric, and if X is a proper geodesic metric space on which G acts properly discontinuously and cocompactly by isometries, then G is quasi-isometric to X .

A partial converse to this result is the *quasi-action principle* which says that if G is a finitely generated group with the word metric and X is a metric space quasi-isometric to G then there is a cobounded, proper quasi-action of G on X ; the constants for this action depend only on the quasi-isometry constants between G and X .

The quasi-action principle motivates the following question, which is a common point of departure for many quasi-isometric rigidity problems:

- Given a proper, cobounded quasi-action of a group G on a metric space X , when can we get some action of G on X ?

Partial information about this question can sometimes be obtained from the following result (see e.g. [KL97]):

Proposition 5 (Coboundedness Principle). *Suppose that a finitely generated G quasi-acts properly and coboundedly on a metric space X . Let \mathcal{H} be a collection of subsets of X which satisfies the following properties: the elements of \mathcal{H} are pairwise coarsely inequivalent in X ; there exists $A \geq 0$ such that for each $g \in G$, $H \in \mathcal{H}$ there is an $H' \in \mathcal{H}$ such that $g \cdot H =_c H' [A]$; and every metric ball in X intersects at most finitely many elements of \mathcal{H} . Then for each $H \in \mathcal{H}$, the stabilizer subgroup $\text{Stab}_G(H) = \{g \in G \mid g \cdot H =_c H\}$ quasi-acts properly and coboundedly on H .*

We will need some coarse algebraic topology. This subject originated in [FS96], by applying Alexander Duality with naturality to fundamental groups of closed aspherical manifolds. These ideas were generalized in [KK99] to work for a general class of spaces, the *coarse PD(n) spaces* X . These are “uniformly acyclic” simplicial complexes which satisfy a coarse, uniform version of the Poincaré duality property of \mathbf{R}^n : there exist chain maps $C_*(X) \xrightarrow{D} C_c^{n-*}(X)$ and $C_c^{n-*}(X) \xrightarrow{\overline{D}} C_*(X)$ with “uniform distortion” such that $D \circ \overline{D}$ and $\overline{D} \circ D$ are “uniformly chain homotopic” to the identity; see [KK99] for the full development. A coarse PD(n) group G is one which acts properly and coboundedly on some coarse PD(n) space.

A subset H of a metric space X is *deep* if for each r there exists $x \in X$ such that the ball in X around x of radius r is a subset of H .

Proposition 6 (Coarse Jordan Separation). *If X is a coarse PD(n) space and $S \subset X$ is a uniformly embedded coarse PD($n - 1$) space, then for sufficiently large A there are exactly two deep components of $X - N_A(S)$. The coarse intersection of these two components is S .*

For fundamental groups of closed, aspherical n -manifolds, this result with a weaker conclusion of “at least two deep components” comes from [FS96]. The improvement to “exactly two deep components”, and the generalization to coarse PD(n) spaces, comes from [KK99].

We will state other coarse algebraic topology results as we need them below.

Quasi-actions on trees

Suppose that Γ is a finite type graph of groups, $B \rightarrow \Gamma$ is an associated graph of spaces, and $X \rightarrow T$ is the associated Bass-Serre tree of spaces. We make the

additional assumption that each edge and vertex space of B is compact; for instance, if all edge and vertex groups are finitely presented then we can take the edge and vertex spaces to be presentation complexes. We impose a geodesic metric on B , which lifts to a geodesic metric on X . It follows that $\pi_1\Gamma$ is quasi-isometric to X . If G is any finitely generated group quasi-isometric to $\pi_1\Gamma$, it follows that G is also quasi-isometric to X , and therefore G has a cobounded, proper quasi-action on X . In this situation, the motivating question is:

- Does the quasi-action of G on X coarsely respect the vertex and edge spaces of X ?

While this question is somewhat vague, there are several distinct ways to make it more precise. Here is the most rigid possible behavior:

- **Tree rigidity** There is an action of G on T such that for each $g \in G$, $v \in \mathcal{V}(T)$, and $e \in \mathcal{E}(T)$ we have $g \cdot X(v) =_c X(g \cdot v)$ and $g \cdot X(e) =_c X(g \cdot e)$ (uniformly, i.e. with uniform coarseness constant independent of g, v).

Tree rigidity immediately implies, for example, that G is the fundamental group of a graph of groups whose vertex and edge groups are all quasi-isometric to vertex and edge groups of Γ .

In general, tree rigidity is a lot to expect. Ignoring edges for the moment, here is a sequence of vertex rigidity properties, from weaker to stronger:

- **Weak vertex rigidity** For each $g \in G$, $v \in \mathcal{V}(T)$ there exists $w \in \mathcal{V}(T)$ such that $g \cdot X(v) \subset_c X(w)$ [uniformly].
- **Vertex rigidity** For each $g \in G$, $v \in \mathcal{V}(T)$ there exists $w \in \mathcal{V}(T)$ such that $g \cdot X(v) =_c X(w)$ [uniformly].
- **Vertex rigidity with uniqueness** For each $g \in G$, $v \in \mathcal{V}(T)$ there exists a unique $w \in \mathcal{V}(T)$ such that $g \cdot X(v) =_c X(w)$ [uniformly].

In general these three properties are not equivalent. In certain situations they become equivalent, e.g. when no vertex space is coarsely contained in another, which occurs if and only if every edge-to-vertex group injection has infinite index (a property which fails completely in the context of Theorem 1).

The utility of vertex rigidity with uniqueness, for instance, is that it allows us to define an action of G on $\mathcal{V}(T)$, where $g \cdot v = w$ if and only if $g \cdot X(v) =_c X(w)$. Even then we don't get any action of G on the edges, without knowing something more about edge spaces, such as:

- **Strong edge rigidity** For all edges $e \neq e' \in \mathcal{E}(T)$, $X(e) \not\subset_c X(e')$.

Vertex rigidity with uniqueness, coupled with strong edge rigidity, implies tree rigidity, for if e is an edge with endpoints v_1, v_2 , and if e' is the edge with endpoints $g \cdot v_1, g \cdot v_2$, then we have

$$g \cdot X(e) =_c g \cdot (X(v_1) \cap_c X(v_2)) =_c g \cdot X(v_1) \cap_c g \cdot X(v_2) =_c X(g \cdot v_1) \cap_c X(g \cdot v_2) =_c X(e')$$

and so by setting $g \cdot e = e'$ we obtain a well-defined action of G on T satisfying tree rigidity.

Example By the work of Kapovich–Leeb [KL97], the torus decomposition of a non-SOLV Haken 3-manifold satisfies tree rigidity. The heart of their work is a proof of weak vertex rigidity, using asymptotic cones. Tree rigidity follows easily from that, because the torus decomposition has infinite index edge-to-vertex group injections, and obviously satisfies strong edge rigidity.

Homogeneous graphs of groups: proof of Theorem 1

A graph of groups is *geometrically homogeneous* if it satisfies any of the following equivalent statements: each edge-to-vertex injection is a quasi-isometry; each edge-to-vertex injection has finite index image; the Bass-Serre tree has bounded valence. It follows that the edge and vertex groups are all quasi-isometric.

Here's the first step in the proof of Theorem 1:

Theorem 7. *Suppose Γ, Γ' are geometrically homogeneous graphs of coarse PD groups with bushy Bass-Serre trees T, T' , and trees of spaces X, X' . If $h: X \rightarrow X'$ is a quasi-isometry then h respects vertex spaces. More precisely, for each K, C there exists A such that if $h: X \rightarrow X'$ is a K, C quasi-isometry then for each $v \in \mathcal{V}(T)$ there exists $v' \in \mathcal{V}(T')$ such that $h(X(v)) =_c X'(v')$ [A].*

A proof with some extra topological assumptions is found in [FM00b], and that proof generalizes almost word-for-word to coarse PD(n) vertex and edge groups. Applying Theorem 7 to each element of a quasi-action we obtain:

Corollary 8. *With the same notation as in Theorem 7, if H is a finitely generated group quasi-isometric to $\pi_1 \Gamma$ then the quasi-action of H on X satisfies vertex rigidity. It follows that there is a cobounded quasi-action of H on $\mathcal{V}(T)$, with the property that $h \cdot X(v) =_c X(h \cdot v)$ [uniformly].*

The following result is the technical heart of Theorem 1:

Theorem 9. *Let T be a bushy tree of uniformly bounded valence. Let H be a group quasi-acting coboundedly on T . Then the quasi-action of H on T is quasiconjugate to*

a cobounded action of H on a tree T' . That is, there is a quasi-isometry $f: T \rightarrow T'$ such that $h \cdot (f(v)) =_c f(h \cdot v)$ [uniformly].

This combines with Corollary 8 to prove Theorem 1, for one can show first that the tree T' has bounded valence, and so there is a graph of groups Γ' with fundamental group H and Bass-Serre tree T' . Furthermore, the quasi-isometry $H \rightarrow \pi_1(\Gamma)$ takes the vertex and edge stabilizers of the action of H on T' to subsets of $\pi_1\Gamma$ coarsely equivalent to the vertex and edge stabilizers of the action of $\pi_1\Gamma$ on T .

Proof of Theorem 9. The first step of the proof is to construct a vertex set and an action of H . We do not even have a true action of H on $\mathcal{V}(T)$, however. What we do have is an action of H on the space of ends $\text{Ends}(T)$. This can be promoted into an action of H on a new vertex space as follows. Bushiness and bounded valence of T tells us that the space of ends $\text{Ends}(T)$ is a Cantor set. Note that each edge of T determines a partition of $\text{Ends}(T)$ into a pairwise disjoint set of clopens (closed-open subsets). Define a *quasi-edge* of T to be any partition of $\text{Ends}(T)$ into a pair of disjoint clopens. Clearly H acts on the set of quasi-edges of T . If $O \subset \text{Ends}(T)$ is a clopen then the convex hull $\mathcal{H}(O)$ is a subtree of T . If $\{O, O'\}$ is a quasi-edge then the intersection of the 1-neighborhoods of the convex hulls $N_1(\mathcal{H}(O)) \cap N_1(\mathcal{H}(O'))$ is bounded; the diameter of this intersection is defined to be the *quasi-edge constant* of $\{O, O'\}$. Note that a 1-quasi-edge is the same thing as (the partition determined by) a true edge of T . The action of a (K, C) quasi-isometry of T on an A -quasi-edge produces an A' -quasi-edge, where A' depends only on K, C, A . It follows that the H -orbit of a single quasi-edge has uniform quasi-edge constant.

We therefore *define* a vertex set Y^0 to be the H -orbit of some arbitrarily chosen quasi-edge of T . It is not hard, using uniformity of the quasi-edge constant, to attach edges to Y^0 in a locally finite, H -equivariant manner, thereby producing a locally finite graph Y^1 on which H acts coboundedly, and a coarse conjugation $T \rightarrow Y^1$.

Unfortunately, Y^1 may not be a tree, i.e. in particular it may not be simply connected. The latter problem can be corrected without too much difficulty by attaching 2-cells to Y^1 in a locally finite, H -equivariant manner, producing a locally finite, simply connected 2-complex Y^2 on which H acts coboundedly; the coarse conjugation $T \rightarrow Y^1$ extends to a coarse conjugation $T \rightarrow Y^2$.

The final step is to get from the 2-complex Y^2 back to a tree. This is accomplished by using tracks in the sense of Dunwoody [Dun85]: a track τ in Y^2 is a locally separating, connected 1-complex in general position with respect to the skeleta of Y^2 . Since Y^2 is simply connected, each track τ separates Y^2 into two components, and τ is *essential* if each component of $Y - \tau$ is unbounded. Using minimal surface ideas as in [Dun85], we construct an H -equivariant family of pairwise disjoint, essential tracks $\{\tau_i\}$ in Y^2 . By a Haken finiteness argument as in [Dun85] one shows that a maximal

such family $\{\tau_i\}$ exists, and that all components of $Y - \bigcup_i \tau_i$ are bounded. The final tree T' has vertex set in 1–1 correspondence with the components of $Y - \bigcup_i \tau_i$, and edge set in 1–1 correspondence with the set $\{\tau_i\}$. \diamond

Theorem 9 has an interesting consequence concerning lattices in locally compact topological groups. Some quasi-isometry classes \mathcal{C} have the nice property that there is a locally compact group H such that each group $G \in \mathcal{C}$ has a discrete, cocompact quotient in H with finite kernel. This is true, for example, when \mathcal{C} is the quasi-isometry class of cocompact lattices in a semisimple Lie group. However, this doesn't work for $\mathcal{C} = \{\text{virtually free of rank } \geq 2\}$:

Corollary 10. *There does not exist a locally compact group H such that every group which is virtually free of rank ≥ 2 has a discrete, cocompact quotient in H with finite kernel.*

Proof. Suppose H exists as stated. Choose a free, discrete, cocompact subgroup $G \subset H$. We may give H a left-invariant metric so that the inclusion of G into H is a quasi-isometry. Since H is quasi-isometric to the free group G , it follows that H is quasi-isometric to a finite valence tree T . The left action of H on itself is quasi-conjugate to a cobounded quasi-action of H on T . Applying Theorem 9 it follows that H has a cobounded quasi-action on a bushy tree T' of bounded valence. It follows that *every* virtually free group G has a cobounded action on T' . Pick a prime p larger than the maximal valence of a vertex of T' . The group $G = \mathbf{Z}/p * \mathbf{Z}/p$ is virtually free, and therefore has discrete cocompact image in H with finite kernel, and so G acts coboundedly on T' . But each of the free factors \mathbf{Z}/p acts trivially on T' , making G act trivially, contradicting coboundedness. \diamond

Inhomogeneous graphs of groups

Theorem 3, and part of the conclusion of Theorem 4, will both follow from more general results about geometrically inhomogeneous graphs of groups. Although as in the homogeneous case we have results in wider contexts, for present purposes we restrict to coarse PD(n) vertex and edge groups.

Suppose that Γ is a graph of coarse PD(n) groups of various dimensions. An n -raft is a connected subgraph of Γ (or of T) of constant dimension n , such that each edge incident to the raft but not contained in the raft has dimension $< n$. Rafts in T are connected lifts of rafts in Γ . Each raft in T is the Bass-Serre tree for the corresponding quotient raft in Γ .

Assuming Γ is reduced, each line-like raft in T is actually a line, and the quotient is a mapping torus raft in Γ . The problem with mapping tori is that they generally fail to satisfy even weak vertex rigidity. For example, a closed hyperbolic 3-manifold fibering

over S^1 has a fundamental group whose quasi-isometry group is all of $\text{QI}(\mathbf{H}^3) = \text{QC}(S^2)$, and weak vertex rigidity fails miserably. A lattice in 3-dimensional SOLV geometry is a mapping torus of \mathbf{Z}^2 , and while vertex rigidity is conjectured to hold [FM00c], the conjecture remains open. For these and other reasons, in all of our further theorems we must avoid line-like rafts in Bass-Serre trees.

Consider an n -dimensional point raft v of the Bass-Serre tree T with associated vertex space $X(v) \subset X$. Let \mathcal{E} be the edge space pattern inside $X(v)$. For each $m < n$ define a subset $\mathcal{E}_m \subset \mathcal{E}$ of m -dimensional edge spaces inside $X(v)$, and let $\mathcal{E}_{[m,m']} = \mathcal{E}_m \cup \mathcal{E}_{m+1} \cup \dots \cup \mathcal{E}_{m'}$. The Coarse Jordan Separation Theorem implies that each $E \in \mathcal{E}_{n-1}$ coarsely separates $X(v)$ into two deep pieces, whose coarse intersection is E ; in this situation we say that a subset of $X(v)$ *crosses* E if it intersects each of the two deep pieces arbitrarily far from E . Define the *crossing graph* of $X(v)$ to be the graph whose vertex set is \mathcal{E}_{n-1} , with an edge between $E, E' \in \mathcal{E}_{n-1}$ if E, E' cross each other or if there exists some element of $\mathcal{E}_{[1,n-2]}$ which crosses both E and E' .

Theorem 11. *Let Γ be a finite, reduced graph of coarse PD groups satisfying the following “raft hypotheses”: the Bass-Serre tree T has no line rafts; and for each point raft v of T , the crossing graph of $X(v)$ is connected or empty. If H is a finitely generated group quasi-isometric to $\pi_1\Gamma$, then there is a finite type, reduced graph of groups Γ' with $H \approx \pi_1\Gamma'$ and with Bass-Serre tree of spaces $X' \rightarrow T'$, and there is a quasi-isometry $f: X' \rightarrow X$ coarsely conjugating the H action on X' to the H -quasi-action on X , such that f coarsely respects rafts and their vertex spaces, and f coarsely respects all edge spaces. That is: for each raft t' of X' there exists a raft t of X such that $f(X'(t')) =_c X(t)$ [uniformly], for each vertex $v' \in t'$ there exists a vertex $v \in t$ such that $f(X'(v')) =_c X(v)$ [uniformly]; and for each edge e' of T' there exists an edge e of T such that $f(X'(e')) =_c X(e)$ [uniformly].*

Although the conclusion makes no mention of vertices $v' \in T'$ not on any raft of T' , such information may be derived as follows: since v' is not on any raft, there exists an edge $e' \subset T'$ incident to v' such that $X'(v') =_c X'(e')$, and so $f(X'(v')) =_c f(X'(e'))$ is coarsely equivalent to some edge space in X . However, counterexamples show that $f(X'(v'))$ might not be coarsely equivalent to any vertex space in X .

Before proving the theorem, we apply it to Theorem 3 and part of Theorem 4.

The hypothesis (*) in Theorem 3 immediately implies the raft hypotheses in Theorem 11, and the conclusion of Theorem 11 immediately implies the conclusion of Theorem 3. Note however that Theorem 11 gives a much stronger conclusion, namely that the new Bass-Serre tree of spaces $X' \rightarrow T'$ maps to original Bass-Serre tree of spaces $X \rightarrow T$, preserving the coarse inclusions among vertex and edge spaces. We'll use this additional information later, combining it with our Abelian Pattern Rigidity Theorem 14 to obtain a stronger rigidity result.

For Theorem 4, the fact that vertex groups have dimension ≥ 3 and edge groups have dimension 1 implies that all rafts are point rafts with empty crossing graph, and so Theorem 11 applies. We conclude that all edge groups of Γ' are quasi-isometric to \mathbf{Z} , and therefore commensurable to \mathbf{Z} . We also conclude that each vertex group $\Gamma'_{v'}$ satisfies one of two possibilities: $\Gamma'_{v'}$ is quasi-isometric and so commensurable to \mathbf{Z} ; or $\Gamma'_{v'}$ is quasi-isometric to some vertex group Γ_v , and so to \mathbf{H}^n , and so $\Gamma'_{v'}$ is weakly commensurable to *some* closed \mathbf{H}^n orbifold group. But in the latter case we get more information: the ambient quasi-isometry $X' \rightarrow X$ takes $X'(v')$ to $X(v)$, coarsely mapping the edge space pattern inside $X'(v')$ to the edge space pattern inside $X(v)$. We'll make this more precise later, and combine it with Schwartz' Geodesic Pattern Rigidity Theorem 13, to get the stronger conclusion of Theorem 4, namely that $\Gamma'_{v'}$ is weakly commensurable to Γ_v itself.

Sketch of proof of Theorem 11

Let N be the maximal dimension of a vertex in T , and define a filtration $T_N \subset \cdots \subset T_{i+1} \subset T_i \subset \cdots \subset T_0 = T$ where T_i is the union of all vertices and edges of dimension $\geq i$. Note that T_N is a disjoint union of N -rafts. There may be lower dimensional rafts as well: any component of T_i which does not contain a component of T_{i+1} is an i -raft. Let Γ_i be the image of T_i in Γ , and let X_i be the inverse image of T_i in X . Thus, each component of X_i maps to a component of T_i , giving a Bass-Serre tree of spaces for the corresponding component of Γ_i .

Let H be a group quasi-isometric to $\pi_1\Gamma$, and so H quasi-acts properly and coboundedly on X . The method of the proof is to work inductively down from the top dimension, replacing the quasi-action of H on the tree of spaces $X_i \rightarrow T_i$ by a true action, starting with $i = N$. The basis step of the induction depends on the following:

Proposition 12 (Vertex rigidity at the top dimension). *The quasi-action of H on X coarsely respects N -rafts and their vertex spaces, that is: for each N -raft $t \subset T$ and each $h \in H$ there exists an N -raft $t' \subset T$ such that $h \cdot X(t) =_c X(t')$ [uniformly], and for each vertex $v \in t$ there exists a vertex $v' \in t'$ such that $h \cdot X(v) =_c X(v')$ [uniformly].*

Once this is shown, applying Proposition 5 to the collection of N -rafts we conclude that the stabilizer of each N -raft quasi-acts coboundedly and properly, and then applying Theorem 1 we may quasi-conjugate the action of $\text{Stab}(T)$ on each N -raft T to a true action of $\text{Stab}(T)$ on a new tree T' . Thus we establish the basis step for the inductive proof of Theorem 11.

Proof of Proposition 12. We shall prove:

(*) There exists A such that for each $v \in T_N$ and each $h \in H$ there is an N -raft t with $h \cdot X(v) \subset_c X'(t)$ $[A]$.

To see why this suffices, consider a raft t of T_N and two vertices $v_1, v_2 \in \mathcal{V}(t)$, and so $X(v_1) =_c X(v_2)$. Applying (*) we get $h \cdot X(v_1) \subset_c X'(t_1)$, $h \cdot X(v_2) \subset_c X'(t_2)$ for N -rafts t_1, t_2 of T , and we want to verify that $t_1 = t_2$. Since $X(v_1) =_c X(v_2)$ it follows that the coarse intersection of $X'(t_1)$ and $X'(t_2)$ coarsely contains a coarse PD(N) space, namely $h \cdot X(v_1) =_c h \cdot X(v_2)$. If $t_1 \neq t_2$ then the coarse intersection of $X'(t_1)$ and $X'(t_2)$ is coarsely equivalent to some edge space of lower dimension, but a coarse PD(n) space with $n < N$ cannot contain a uniformly embedded copy of a coarse PD(N) space. It follows that $t_1 = t_2$ and $h \cdot X(t) \subset_c X(t_1)$. Similarly $h^{-1} \cdot X(t_1) \subset_c X(t')$ for some N -raft t' , and so $X(t') \subset_c X(t)$; but this is only possible if $t = t'$. This shows that H coarsely respects N -rafts.

If (*) is not true then, taking counterexamples for larger and larger values of A , using coboundedness of the isometry group of $X \rightarrow T$, and passing to a limit, we obtain a quasi-isometry $h: X \rightarrow X$, a vertex $v \in \mathcal{V}(T_N)$ and an edge $e \in \mathcal{E}(T) - \mathcal{E}(T_N)$, such that $h(X(v))$ intersects both components of $X - X(e)$ arbitrarily deeply. It follows that for sufficiently large A , the subset

$$S = h^{-1}(N_A(X(e))) \cap X(v)$$

coarsely separates $X(v)$ into at least two deep components. But the set S , with metric restricted from $X(v)$, is uniformly equivalent to a subset of $X(e)$ with restricted metric.

When $X(e)$ has dimension $\leq N - 2$ we obtain a contradiction using arguments of coarse algebraic topology: a coarse PD(N) space cannot be coarsely separated by a subset which is uniformly equivalent to a subset of a coarse PD space of dimension $\leq N - 2$.

If $X(e)$ is of dimension $= N - 1$ then in fact $\pi(h(S)) =_c X(e)$: otherwise, a subset of the coarse PD($N - 1$) space $X(e)$ which is not coarsely equivalent to all of $X(e)$ would embed uniformly in coarse PD(N) space $X(v)$, coarsely separating $X(v)$, and that is impossible. We therefore have $h(S) =_c X(e)$ in X . This shows that S is a coarse PD($N - 1$) space uniformly embedded in the coarse PD(N) space $X(v)$, and so $X(v) - S$ has exactly two deep components each coarsely containing S , each contained in a separate deep component of $X - S$.

Now the argument breaks into cases.

Case 1: v is contained in a bushy raft t . In this case the coarse PD($N - 1$) space S coarsely separates $X(t)$, a tree of PD(N) spaces, and that is clearly impossible.

Case 2: v is a point raft. By hypothesis, the crossing graph of $X(v)$ is either connected or empty.

Case 2a: The crossing graph of $X(v)$ is connected. Using connectedness of the crossing graph together with some coarse separation arguments, one shows that one of the deep components of $X(v) - S$ coarsely contains the union of all codimension-1 edge spaces inside $X(v)$. But this is absurd, because coboundedness of the $\pi_1\Gamma$ stabilizer subgroup of $X(v)$ shows that the union of incident codimension-1 edge spaces intersects every deep subset of $X(v)$.

Case 2b: The crossing graph of $X(v)$ is empty. Each edge incident to v therefore has dimension $\leq N - 2$, and it follows that the inclusion of S in X has the following “coarse Jordan separation property”: for all sufficiently large $A \geq 0$ there are *exactly* two deep components of $X - N_A(S)$ which coarsely contain S . The inclusion of $X(e) =_c h(S)$ into X therefore has the same property: for all sufficiently large A there are exactly two deep components of $X - N_A(X(e))$ which coarsely contain $X(e)$.

Let T_e^* be the subtree of T spanned by all edges in T whose edge space is coarsely equivalent to $X(e)$; we think of T_e^* as the “edge raft” containing e , although a priori T_e^* can have N -dimensional vertices and edges. But by using the coarse Jordan separation property for $X(e)$ one shows that T_e^* contains at most one N -dimensional vertex of T . Moreover, T_e^* cannot contain exactly one N -dimensional vertex, for then one would be able to find an $N - 1$ dimensional valence 1 vertex of T_e^* which would violate irreducibility of the Bass-Serre tree T . It follows that T_e^* in fact consists entirely of $N - 1$ dimensional vertices and edges, and so is an $N - 1$ raft. T_e^* cannot be a bounded raft, for it has at least one edge, namely e , and again that would violate irreducibility. T_e^* cannot be a line raft, by hypothesis. Finally, it cannot be a bushy raft, because that would violate the coarse Jordan separation property for $X(e)$: for larger and larger A , the number of deep components of $X - N_A(X(e))$ coarsely containing $X(e)$ would approach infinity. \diamond

We now continue the proof of Theorem 11 by induction down the dimension. Suppose by induction that we have altered X and T down to dimension $n + 1$, producing a filtered tree of spaces

$$\begin{array}{ccccccc} X'_N & \xrightarrow{\subset} & \cdots & \xrightarrow{\subset} & X'_{n+1} & \xrightarrow{\subset} & X_n & \xrightarrow{\subset} & \cdots & \xrightarrow{\subset} & X_0 \\ \downarrow & & & & \downarrow & & \downarrow & & & & \downarrow \\ T'_N & \xrightarrow{\subset} & \cdots & \xrightarrow{\subset} & T'_{n+1} & \xrightarrow{\subset} & T_n & \xrightarrow{\subset} & \cdots & \xrightarrow{\subset} & T_0 \end{array}$$

so that H quasi-acts properly and coboundedly on X_0 , restricting to a true action on $X'_k \rightarrow T'_k$, $N \geq k \geq n + 1$, and we have an H -quasiconjugation back to the original tree of spaces, restricting to the identity on $X_0 - X'_{n+1}$, and satisfying the conclusions of Theorem 11 on X'_{n+1} .

The arguments of the basis step can be applied to any component of T_n which is an n -raft.

Consider now an edge e of $T_n - T'_{n+1}$ which does not lie on an n -raft, and an element $h \in H$; we study the image $h \cdot X(e)$. Using irreducibility of the original tree of spaces it follows that there are two vertices $v, w \in T'_{n+1}$ such that $X(e)$ is a coarse intersection of $X'(v)$ and $X'(w)$, with constants independent of e ; moreover, v, w can be chosen to have a distance bounded above in T'_{n+1} independent of e . Now we apply the inductive hypothesis to $h \cdot X(v), h \cdot X(w)$, splitting into subcases depending on whether v, w lie on rafts.

Case 1: Suppose v, w do lie on rafts. By induction, there exist vertices v', w' such $h \cdot X(v) =_c X(v')$, $h \cdot X(w) =_c X(w')$. It follows that $h \cdot X(e)$ is a coarse intersection of $X(v')$ and $X(w')$. Let e_1, \dots, e_K be the simple edge path in T_0 connecting v' to w' ; K is bounded independent of v, w , depending only on the quasi-isometry constants of h . The coarse intersection of $X(v')$ and $X(w')$ equals the coarse intersection of $X(e_1), \dots, X(e_K)$. This implies that $h \cdot X(e)$ is a coarse intersection of $X(e_1), \dots, X(e_K)$, and in particular $h \cdot X(e)$ is coarsely contained in each of $X(e_1), \dots, X(e_K)$.

The first consequence of this is that the dimensions of e_1, \dots, e_K are greater than or equal to the dimension of e , because a coarse $\text{PD}(n)$ space cannot uniformly embed in a coarse PD space of lower dimension.

The second consequence is that if e_k is n -dimensional then $h \cdot X(e) =_c X(e_k)$, because of “packing”: a uniform embedding of a coarse $\text{PD}(n)$ space in a coarse $\text{PD}(n)$ space must have image coarsely equivalent to the whole space.

To complete the proof in Case 1 it therefore remains to check that the edges e_1, \dots, e_K cannot all have dimension $\geq N + 1$. For this we need the fact that $X(e)$ coarsely separates $X(v)$ and $X(w)$ in X , and it follows that $h \cdot X(e)$ coarsely separates $X(v')$ and $X(w')$ in X . But if e_1, \dots, e_K all have dimension $\geq N + 1$ then the coarse $\text{PD}(n)$ space $h \cdot X(e)$ cannot coarsely separate $X(v')$ and $X(w')$.

Remaining cases: If, say, v lies on a raft and w does not, then w is incident to an edge e_1 of the same dimension. Applying induction, $h \cdot X(v) =_c X(v')$ for some raft vertex v' , and $h \cdot X(w) =_c h \cdot X(e_1) =_c X(e')$ for some edge e' . Now connect v' and e' by a simple edge path and repeat the arguments of Case 1. The other cases are similar.

We have shown that the quasi-action of H coarsely respects n -dimensional edge spaces. It remains to attach additional edges to the H -forest T'_{n+1} to make an H -forest T'_n , so that a newly attached edge e' between vertices $v', w' \in T'_{n+1}$ has an coarse $\text{PD}(n)$ edge space $X'(e') \subset X'_n$ taken coarsely to some coarse $\text{PD}(n)$ edge space $X(e) \subset X_n$. The construction of T'_n is a relative version of the construction in

the basis step, which itself is adapted from the proof of the Homogeneous Theorem 1. First one chooses any n -dimensional edge e of T_n , between vertices $v', w' \in T'_{n+1}$. Then for each $h \in H$ one attaches an edge $h \cdot e$ between $h \cdot v'$ and $h \cdot w'$. The result is not a forest, but it is quasi-isometrically identified with the forest T_n . Now attach 2-cells in an H -equivariant manner, with finitely many H -orbits, so that each of the edges of a given 2-cell have edge spaces in the same coarse equivalent class. Now we have an H -complex each of whose components is simply connected. Apply tracks to get an H -forest containing T'_{n+1} as a subforest. Each new edge (resp. vertex) of this forest has an n -dimensional edge space (resp. vertex space) mapping back to a n -dimensional edge space of X_n .

Pattern rigidity

Suppose we are in the setting of Theorem 11. Let v' be a point raft of T' and v the corresponding point raft of T . Note that as a consequence of Theorem 11, the quasi-isometry $X'(v') \rightarrow X(v)$ takes the edge space pattern inside $X'(v')$ coarsely to the edge space pattern inside $X(v)$. This information can be used to strengthen applications of Theorem 11, by applying “pattern rigidity” results.

For example, in the setting of Theorem 4 we have the following theorem of R. Schwartz [Sch97]:

Theorem 13 (Geodesic Pattern Rigidity in \mathbf{H}^n). *Let G be a discrete, cocompact group of isometries of \mathbf{H}^n , $n \geq 3$. Let A be a nonempty, G -equivariant set of geodesics in \mathbf{H}^n , with finitely many G -orbits. Let H be a group and let $H \times \mathbf{H}^n \xrightarrow{\phi} \mathbf{H}^n$ be a cobounded, proper quasi-action which coarsely respects A : for each $h \in H$ and $a \in A$ there exists $a' \in A$ such that $\phi(h, a) =_c a'$. Then there is an isometric action $\psi: H \rightarrow \text{Isom}(\mathbf{H}^n)$ with finite kernel which strictly respects A , such that ψ is a bounded distance from ϕ , i.e. $\sup_{h,x} d(\phi(h, x), \psi(h, x)) < \infty$.*

In this setting of this theorem, let $\text{Isom}(\mathbf{H}^n, A)$ be the set of isometries of \mathbf{H}^n respecting A . This is a discrete, cocompact group of isometries, containing G and $H/\text{Ker}(\psi)$ as finite index subgroups. It follows that G and $H/\text{Ker}(\psi)$ are commensurable, by a commensuration taking the A -stabilizers in G to the A -stabilizers of $H/\text{Ker}(\psi)$. Combined with the discussion above, this completes the proof of Theorem 4.

We can get even stronger conclusions under stronger hypotheses. For instance, consider a graph of groups Γ as in Theorem 4. Recall that in a discrete, cocompact group of isometries of \mathbf{H}^n , the set of loxodromic axis stabilizers is identical to the set of maximal, virtually cyclic subgroups. Suppose that we make the following additional assumption:

- Each edge-to-vertex injection $\xi_\eta: \Gamma_e \rightarrow \Gamma_{v(\eta)}$ has image equal to a maximal virtually cyclic subgroup of $\Gamma_{v(\eta)}$, and two distinct edge-to-vertex injections into $\Gamma_{v(\eta)}$ have distinct images.

This implies that for any vertex v of the Bass-Serre tree T , distinct incident edge spaces inside $X(v)$ are all coarsely inequivalent in $X(v)$. The Bestvina-Feighn Combination Theorem [BF92] implies that $\pi_1\Gamma$ is word hyperbolic.

With this additional assumption, Theorem 11 implies that for any group H quasi-isometric to $\pi_1\Gamma$, the proper, cobounded quasi-action of H on X satisfies tree rigidity. Combining this with Geodesic Pattern Rigidity, it follows that H is weakly commensurable to $\pi_1\Gamma$.

In fact, what this argument shows, under the additional assumption, is that the group $\pi_1\Gamma$ has finite index in its quasi-isometry group $\text{QI}(\pi_1\Gamma)$; this is the strongest form of quasi-isometric rigidity. We also obtain a computation the abstract commensurator of $\pi_1\Gamma$: it is isomorphic to $\text{QI}(\pi_1\Gamma)$, which is isomorphic to the full isometry group of the tree of spaces X .

Finally we turn to abelian pattern rigidity and a strengthening of Theorem 3.

Theorem 14 (Abelian Pattern Rigidity). *Suppose that $V_1, \dots, V_K \subset \mathbf{E}^n$ and $W_1, \dots, W_K \subset \mathbf{E}^n$ are affine foliations. Suppose that there exists a quasi-isometry $f: \mathbf{E}^n \rightarrow \mathbf{E}^n$ which maps W_k coarsely to V_k , for each $k = 1, \dots, K$. Then there exists a linear isomorphism $F: \mathbf{E}^n \rightarrow \mathbf{E}^n$ such that $F(W_k) = V_k$ for each k .*

Proof. By passing to asymptotic cones we replace f with a bilipschitz homeomorphism taking V_k to W_k for each k . Applying the Rademacher Theorem, at almost any point x the derivative $F = D_x f$ gives the desired conclusion. \diamond

If $S(V)$ is the linear subspace of \mathbf{E}^n parallel to the leaves of an affine foliation V , then a pattern of affine foliations V_1, \dots, V_K induces a pattern of linear subspaces $S(V_1), \dots, S(V_K) \subset \mathbf{E}^n$, which in turn induces a pattern of projective subspaces $\mathbf{PS}(V_1), \dots, \mathbf{PS}(V_K) \subset \mathbf{P}^{n-1}$, called the *projective pattern* associated to V_1, \dots, V_K . The Abelian Pattern Rigidity Theorem 14 shows that the associated projective pattern is a quasi-isometry invariant of patterns of affine foliations in \mathbf{E}^n . For example, four distinct 1-dimensional affine foliations of \mathbf{E}^2 have an associated projective pattern of four distinct points in \mathbf{P}^1 . The moduli space of such projective patterns is 1-dimensional, parameterized by the cross-ratio, and so the cross-ratio is a pattern preserving quasi-isometry invariant.

We can strengthen Theorem 3 as follows. Suppose that Γ is a reduced graph of abelian groups as in Theorem 3, with Bass-Serre tree of spaces $X \rightarrow T$. Recall that

the hypotheses of Theorem 3 imply that each vertex of T is a raft. Our strengthened conclusions say that projective patterns associated to edge space patterns inside vertex spaces of X are quasi-isometrically rigid. Here are the details.

Let H be a finitely generated group quasi-isometric to $\pi_1\Gamma$. The proof of Theorem 3 gives a graph of groups Γ' with Bass-Serre tree of spaces $X' \rightarrow T'$, such that $H \approx \pi_1\Gamma'$, together with a quasi-isometry $\phi: X' \rightarrow X$, such that for each edge e' of T' there is an edge e of T with $\phi(X(e)) =_c X'(e')$, and for each vertex v' of T' , either v' is a raft of T' and $\phi(X'(v')) =_c X(v)$ for some vertex v of T , or v' is not a raft of T' and so v' has an incident edge e' with $X'(v') =_c X'(e')$.

Consider a raft vertex v' of T' and the corresponding vertex v of T . Composing the map $X(v') \xrightarrow{\phi} X'$ with the closest point projection to $X(v)$ gives a quasi-isometry still denoted $\phi: X'(v') \rightarrow X(v)$. Moreover, this quasi-isometry takes the edge space pattern inside $X'(v') \approx \mathbf{E}^n$ coarsely to the edge space pattern inside $X(v) \approx \mathbf{E}^n$. Applying the Abelian Pattern Rigidity Theorem 14, the projective patterns in \mathbf{P}^{n-1} associated to $X'(v')$ and $X(v)$ are projectively equivalent.

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